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CONVOLUTION METHODS FOR MATHEMATICAL PROBLEMS IN BIOMETRICS

by

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Estimation of the impostor probability density function from the inter-template and template-sample probability density functions involves a multi-dimensional convolution of the latter density functions. We assume isotropy of the probability distribution functions and use the Fourier transform to express the convolution as a one-dimensional integral with a kernel given by a spherical Bessel function.

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Convolution Methods for Mathematical Problems in Biometrics

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1.Introduction. The problem we shall investigate can be formulated in the following way, due to Peter Bickel [1]: For $M+1$ individuals K measurements $X_{ij}, i = 1, \cdots, M+1, j = 1, \cdots, K$ are made. The X_{ij} are assumed to be vectors in $\mathbf{R}^N$.

We assume, following Bickel [1], that

$$X_{ij} = \mu_i + \varepsilon_{ij},$$

where $\mu_1, \cdots, \mu_{M+1}$ are individual effects vectors from $\mathbf{R}^N$ which are i.i.d. (independently and identically distributed) with a distribution which is spherically symmetric about some μ in $\mathbf{R}^N$. The ε_{ij} are i.i.d. independently of the μ_i and their distribution is spherically symmetric about 0.

The $K(M+1)X_{ij}$s represent K biometric measurements made on $M+1$ individuals; however the measuring device actually records only the data $\mid \bar{X}_i - \bar{X}_{i'} \mid$ and $\mid X_{M+1,j} - \bar{X}_{M+1} \mid$, $1 \le i < i' \le M, 1 \le j \le k$, where $\mid x \mid$ is the length of x and

$$\bar{X}_i = \frac{1}{K} \sum_{j=1}^{K} X_{ij}$$

is the centroid of the K measurements taken on the the ith individual.

Let p be the density of $\mid \bar{X}_1 - \bar{X}_2 \mid$, q the density of $\mid X_{M+1,1} - \bar{X}_{M+1} \mid$, and r the density of $\mid X_{M+1,1} - \bar{X}_1 \mid$. The basic problem is to estimate r given estimates $\hat{p}, \hat{q}$ for p and q. As p and q are densities on $[0, \infty)$, the problem of estimating $\hat{p}$ and $\hat{q}$ is a standard problem in estimation theory which we do not consider further.

2. Bickel's Approach. What is the relationship between p, q, and r? Following Bickel [1], we write

$$X_{M+1,1} - \bar{X}_1 = \varepsilon_{M+1,1} + \mu_{M+1} - \mu_1 - \bar{\varepsilon}_1, \tag{1}$$

$$\bar{X}_{M+1} - \bar{X}_1 = \bar{\varepsilon}_{M+1} + \mu_{M+1} - \mu_1 - \bar{\varepsilon}_1, \tag{2}$$

$$X_{M+1,1} - \bar{X}_{M+1} = \varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}. \tag{3}$$

Note that addition of equations (2) and (3) yields equation (1). Further, the quantities on the left sides of (2) and (3 have spherically symmetric distributions. As the lengths of the left sides of (2) and (3) have densities p and q respectively, and the length of the left side of (1) is r, it seems that what is required is a formula for the density of the length of the convolution of two spherically symmetric (about 0) distributions given the densities of their lengths. However, as Bickel pointed out in [1], this is not quite correct since the use of the convolution assumes the independence of the densities p and q, and these two densities are not generally independent, since in the right sides of (2) and (3) the terms $\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}$ and $\bar{\varepsilon}_{M+1}$ are only uncorrelated, and not in general independent unless ε_{11} has a Gaussian distribution.

However, for large K, we follow Bickel's argument in [1] to show that the terms $\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1}$ and $\bar{\varepsilon}_{M+1}$ are independent. Let

$$Z_K = (K-1)^{-\frac{1}{2}} \sum_{j=2}^{K} \varepsilon_{M+1,j}, \tag{4}$$

and

$$U_K = \varepsilon_{M+1,1}(1 - \frac{1}{K}). \tag{5}$$

Then, Z_K and U_K are independent and, if K is large,

$$\varepsilon_{M+1,1} - \bar{\varepsilon}_{M+1} = U_K - \frac{(K-1)^{\frac{1}{2}}}{K} Z_K \approx U_K \tag{6}$$

and

$$K^{\frac{1}{2}} \bar{\varepsilon}_{M+1} = \left(\frac{K-1}{K}\right)^{\frac{1}{2}} Z_K + U_K \frac{K^{\frac{1}{2}}}{(K-1)} \approx Z_K. \tag{7}$$

Thus, for large K, to a first approximation it is possible to ignore the dependence between (6) and (7). The terms U_K and Z_K on the right sides of (6) and (7) respectively are the first terms in an asymptotic expansion for large K of the left sides of those equations.

3. Convolution and Fourier Transform. Let U, V be random vectors in $\mathbf{R}^N$ which are independent and have isotropic distributions with densities

f_1, f_2 respectively. We let g_1, g_2 be the corresponding densities on $\mathbf{R}^+$ of the lengths of U and V, $\mid U \mid, \mid V \mid$. Further, let f be the density of $U+V$ and g be the density of $\mid U+V \mid$. Then by the independence of U, V,

$$f(y) = \int_{\mathbf{R}^N} f_2(y-x) f_1(x) dx. \tag{8}$$

Our interest is to determine a formula for g in terms of g_1 and g_2. To this end, we introduce the Fourier transform. If f is absolutely integrable on $\mathbf{R}^N$, then the Fourier transform of f is defined by

$$\hat{f}(t) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} f(y) e^{it\cdot y} dy, \tag{9}$$

where the N vector $t = (t_1, t_2, \cdots, t_N)$ and $t \cdot y = t_1 y_1 + \cdots t_N y_N$. If $f(y)$ is spherically symmetric, i.e., $f(y)$ is a function of $r = \mid y \mid$ only, say $f(y) = h(r)$, then its Fourier transform $\hat{f}(t)$ is also spherically symmetric; more specifically, we have

$$\hat{f}(t) = \rho^{(2-N)/2} \int_0^\infty r^{N/2} h(r) J_{(N-2)/2}(\rho r) dr, \tag{10}$$

where $\rho = \mid t \mid$ and $J_\nu(r)$ denotes the Bessel function of the first kind of order ν. For a proof of this formula, we refer to Schwartz[2]. (Note that we have introduced the same letter r for $\mid y \mid$ as we used for the density of the length of the left side of (1). From the context, there should be no confusion as to which meaning for r is intended.) The Fourier transform of the convolution in (8) yields

$$\hat{f}(t) = \hat{f}_2(t) \hat{f}_1(t), \tag{11}$$

and combining (10) and (11) gives

$$\hat{f}(t) = \rho^{2-N} \int_0^\infty r^{N/2} h_2(r) J_{(N-2)/2}(\rho r) dr \int_0^\infty r^{N/2} h_1(r) J_{(N-2)/2}(\rho r) dr, \tag{12}$$

where the functions h_1, h_2 in (12) are defined by

$$h_1(r) = f_1(y), \qquad h_2(r) = f_2(y), \tag{13}$$

with $r = \mid y \mid$ since the distributions defined by f_1, f_2 are isotropic.

Now if U is an isotropically distributed N vector with density f on $\mathbf{R}^N$, and g is the corresponding density on $\mathbf{R}^+$ of the length $\mid U \mid$, where $\mid x \mid^2 =$

$\sum_{i=1}^{N} x_i^2, x = (x_1, \cdots x_N)$, then the relationship between the densities f and g is given by

$$f(y) = \mid y \mid^{-(N-1)} c_N^{-1} g(\mid y \mid), \tag{14}$$

where c_N, the surface 'area' of the unit sphere in $\mathbf{R}^N$, is

$$c_N = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}. \tag{15}$$

Hence, upon substituting

$$\begin{aligned} h_1(r) = f_1(y) &= r^{-(N-1)} c_N^{-1} g_1(r), \\ h_2(r) = f_1(y) &= r^{-(N-1)} c_N^{-1} g_2(r), \end{aligned} \tag{16}$$

into (12), we have

$$\hat{f}(t) = \rho^{2-N} c_N^{-2} \int_0^\infty r^{1-N/2} g_2(r) J_{(N-2)/2}(\rho r) dr \int_0^\infty r^{1-N/2} g_1(r) J_{(N-2)/2}(\rho r) dr. \tag{17}$$

If the dimension N of the space the measurement vectors X_{ij} belong to is even, say $N = 2m$, then $J_{(N-2)/2}(\rho r) = J_{m-1}(\rho r)$ is a Bessel function of the first kind of integer order. If N is odd, say $N = 2m + 1$, then $J_{(N-2)/2}(\rho r) = J_{m-1/2}(\rho r)$ is a Bessel function of the first kind of fractional order, and is closely related to the Spherical Bessel function of the first kind $j_n(z)$, defined by

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z). \tag{18}$$

More detailed information about Bessel functions may be in found in Abramowitz and Stegun [3].

4. Convergence of the Integrals. We now discuss convergence of the integrals in (17). Both integrals are functions of the variable $\rho = \mid t \mid$, hence $\hat{f}(t)$ is a spherically symmetric function of the transform variable t. This means that the inverse Fourier transform of $\hat{f}(t)$, i.e. $f(y)$, can also be obtained as a one-dimensional integral with a Bessel function kernel.

For fixed ρ and small r, $J_{(N-2)/2}(\rho r) = O(r^{(N-2)/2})$. Since we expect $g_i^{(j)}(0) = 0$ for $i = 1, 2$ for $0 \leq j \leq N-1$ (see Bickel [1]), it follows that both integrals in (17) are convergent at the lower limit 0. At the upper limit ∞, $J_{(N-2)/2}(\rho r) = O(r^{-1/2})$, and this by itself will not be enough to make the integral convergent. However the term $r^{1-N/2}$ will also make the integrals converge at the upper limit if N is suffciently large. In practice the densities g_1, g_2 also tend to zero sufficently rapidly to make the integrals converge at the upper limit. If these densities have compact support (i.e., are zero outside of a closed bounded subset of $\mathbf{R}^+$), then the integrals in (17) no longer have infinite upper limits. The integrals in (17) can be evaluated accurately and efficiently by standard numerical quadrature methods.

5. Inversion. Note that the right side of (17) is a function of $\rho = | t |$ only. Hence $\hat{f}(t)$ is spherically symmetric. Let $\hat{f}(t) = G(\rho)$. The inverse Fourier transform of $\hat{f}(t)$, i.e., $f(y)$ from (9), is defined by

$$f(y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} \hat{f}(t) e^{-it\cdot y} dt. \tag{19}$$

It follows by analogy with (10) that

$$f(y) = r^{(2-N)/2} \int_0^\infty \rho^{N/2} G(\rho) J_{(N-2)/2}(\rho r) d\rho, \tag{20}$$

where $r = | y |$. The relationship (14) between the density f of an isotropically distributed N vector and the corresponding density g of its length then implies

$$g(r) = r^{N/2} c_N \int_0^\infty \rho^{N/2} G(\rho) J_{(N-2)/2}(\rho r) d\rho \tag{21}$$

where $G(\rho) = \hat{f}(t)$ is given by (17). Numerical evaluation of the integral in (21) proceeds similarly to the integrals in (17). With g_1, g_2 taken as p, q introduced at the end of section 1, and g taken as r (the density function for $| X_{M+1,1} - \bar{X}_1 |$, *not* $| y |$), (17) and (21) together give the density r in terms of p and q.

6. Conclusions. We have shown that for large K, the left sides of (2) and (3) are approximately independent. Under this approximation, we derived an expression for the density r of the length of the left side of (1) in terms of the densities p, q of the lengths of the left sides of (2) and (3). This result is contained in equations (17) and (21) of the previous section.

Future work could include determining corrections to the above result for a finite number of measurements K, and practical numerical implementation of the above result.

References

1.) NSA SAG Problem 97-2-1 Solution, Peter Bickel.
2.) *Mathematics for Physical Sciences*, L. Schwartz, Addison-Wesley, Reading, MA, 1966, pp. 201-203.
3.) *Handbook of Mathematical Functions*, A. Abramowitz and I. Stegun, National Bureau of Standards, 1964.

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